congrats on exam!
(contact me with questions)

Math 4200
Monday November 16
4.3 Introduction to integral applications of the residue theorem.

With modern technology that can compute definite and indefinite integrals way beyond Calculus, this topic may seem a little out of date. But it turns out there are integrals that are interesting and computable via the Residue Theorem. We'll see some representative examples today, selecting from the large collection in section 4.3. By Friday we'll see magic formulas for certain types of infinite series (section 4.4). These techniques also lead to interesting infinite sum and infinite product formula for various meromorphic functions on $\mathbb{C}$. Infinite product formulas are related to infinite sum formulas via the logarithm, and some of these infinite products are related to the gamma function and the Riemann Zeta function (chapter 7). In a completely different direction of ideas, there is a contour integral formula for the inverse Laplace transform, 8.1-8.2, and contour integrals are used to understand aspects of the Fourier transforms and inverse transforms as well.

This week: 4.3-4.4 •
Thanksgiving week 5.1-5.2 •
Week after TG MW - presentations

Announcements:

reminder: HW for Friday November 20
4.3: 1, 2, 4, 6, 10, 14, 17, 20ab.
There are a lot of good worked examples in the text. In problem 6 you may use entry #5 on the Definite integral table 4.3.1, page 296. The text explains why this table entry is true on pages 289-293 and summarizes it as Proposition 4.3.16. We'll also discuss related examples today and Wednesday.

4.3: Applications of contour integration to interesting integrals from real variables.

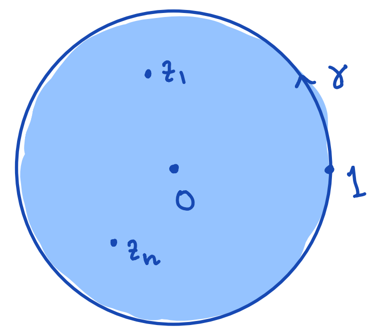
table entry 4 (page 296)

$$\boxed{\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta}$$

e.g. $\dfrac{\cos^2\theta}{(1+\sin^2\theta)^{10}}$

where f is any rational function of $\cos(\theta)$, $\sin(\theta)$, or more generally any function $f(z, w)$ that is analytic in z and w for $|z|,|w| \le 1$, except for isolated singularities.This can be re-expressed as contour integral around the unit circle, and then evaluated using the Residue Theorem: If

- $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ on unit circle.



then working to covert everything into "z" expressions:

$$\frac{1}{z} = e^{-i\theta}$$

$$\left.\begin{aligned} \cos(\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin(\theta) &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right) \end{aligned}\right\} \text{on unit circle.}$$

$$dz = i\, e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}.$$

So

via $z = e^{i\theta}$ ← compute this

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} f\Big(\underbrace{\frac{1}{2}\left(z + \frac{1}{z}\right)}_{\cos\theta}, \underbrace{\frac{1}{2i}\left(z - \frac{1}{z}\right)}_{\sin\theta}\Big) \underbrace{\frac{dz}{iz}}_{d\theta}$$

Your homework problems 4.3.10, 4.3.20a are in this vein.

Example Show

$$\underbrace{\frac{1}{2}\int_0^{2\pi} \cos^4\theta\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \cos^4\theta\, d\theta}_{\text{integrand is } 2\pi\text{-periodic.}} \underbrace{=}_{\text{even integrand}} \int_0^{\pi} \cos^4\theta\, d\theta = \frac{3}{8}\pi.$$

$$\frac{1}{2}\int_0^{2\pi} \cos^4\theta\, d\theta = \frac{1}{2}\int_{|z|=1} \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^4 \frac{dz}{iz}$$

$$= \frac{1}{2}\frac{1}{i}\frac{1}{2^4}\int_{|z|} \frac{1}{z}\left(z + \frac{1}{z}\right)^4 dz$$

$$\frac{1}{z}\left[z^4 + 4\frac{z^3}{z} + 6\frac{z^2}{z^2} + 4\frac{z}{z^3} + \frac{1}{z^4}\right]$$

```
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
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which is the $\frac{1}{z}$ coeff? 6 Res(mess; 0)

Res Thm

$$= \frac{1}{\cancel{2}}\frac{1}{\cancel{i}}\frac{1}{2^4}\cancel{2}\pi\cancel{i}\cdot 6$$

$$= \frac{3}{8}\pi \quad ✓$$

table entries 1 and 2

Theorem If $f(x)$ is the restriction to the real line of function $f(z)$ which is analytic on all of $\mathbb{C}$ except for a finite number of isolated singularities, none of which occur on the real line; and if for large $|z|$ there is a uniform modulus bound

implies conv.

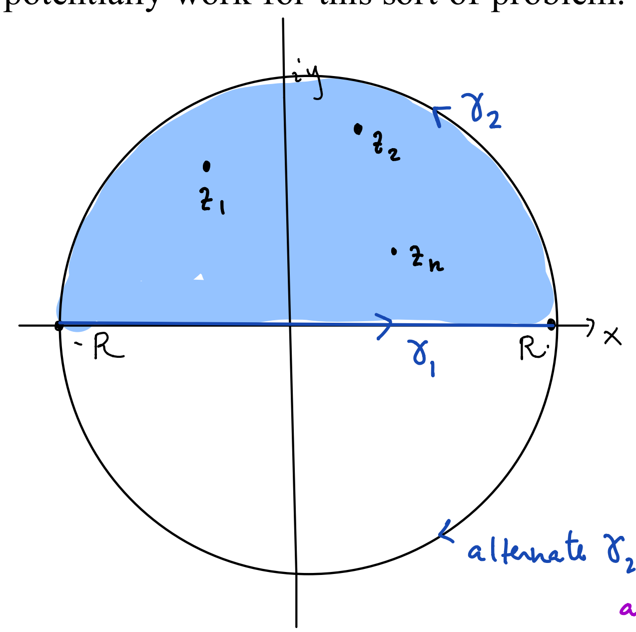
$$\boxed{|f(z)| \le \frac{M}{|z|^2}}$$

in the closed upper half plane $\{z \mid \operatorname{Re}(z) \ge 0\}$(alternately the lower half plane), then

$$\boxed{\int_{-\infty}^{\infty} f(x)\ dx} = 2\pi i \sum \{\text{residues of } f \text{ in the upper half plane}\}$$

- (alternately $\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum \{\text{residues of } f \text{ in the lower half plane}\}$)

Hint: Consider $\gamma_R = \gamma_1 + \gamma_2$, apply the Residue Theorem, and let $R \to \infty$. Make good estimates. Either choice of contour (upper semi-circle, or lower semi-circle) can potentially work for this sort of problem.



$$\int_{\gamma_R} f(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} = \underbrace{\int_{-R}^{R} f(x)\,dx}_{I_1} + \underbrace{\int f(z)\,dz}_{I_2}$$

$z = x$, $dz = dx$ $\qquad$ $|z| = R$, $\operatorname{Im}(z) \ge 0$

as soon as R large enough to contain all sings of f in upper half plane,

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \sum_{\operatorname{Im}(z_k) > 0} \operatorname{Res}(f; z_k)$$

Residue thm.

Now, let $R \to \infty$

$$I_1 \to \int_{-\infty}^{\infty} f(x)\,dx$$

because if $|f(x)| \le \frac{C}{|x|^p}$ $|x|$ large, $p > 1$ integrals converge absolutely so converge.

as $R \to \infty$, $|I_2| \le \int_{\gamma_2} \underbrace{|f(z)|}_{\le \frac{M}{|z|^2}} |dz| \le \frac{M}{R^2} \pi R = \frac{M\pi}{R} \to 0$ as $R \to \infty$.

$\therefore$ result holds.

Example (Relates to homework problem 4.3.4) Compute

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^4+1}\,dx = \int_0^{\infty}\frac{1}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$$

$$\| $$

$$\frac{1}{2}\cdot 2\pi i\left(\sum_{z_k \text{ in upper half plane}} res(f;z_k)\right)$$

$f(z)=\dfrac{1}{z^4+1}$

for $|z|$ large

$|f(z)| \le \dfrac{2}{|z|^4} \le \dfrac{2}{|z|^2}$.

$\left(\dfrac{1}{|z^4+1|} \le \dfrac{1}{|z|^4-1} \right.$

$\left. |z^4+1| \ge |z^4|-1 \quad (|z|\ge 1) \right)$ Reverse Tri Ineq

roots of $z^4=-1$ in upper half plane.

$z=re^{i\theta}$ $\quad z^4 = r^4 e^{i4\theta} = 1e^{i\pi}$ $\quad r^4=1$, $r=1$

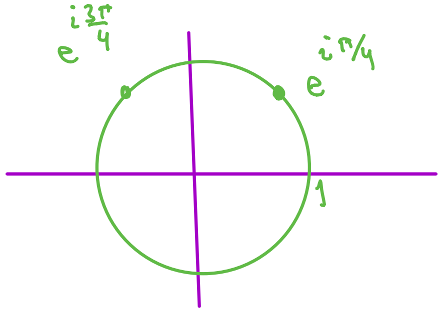
$4\theta = \pi + 2\pi k$

$\theta = \frac{\pi}{4} + \frac{\pi}{2}k$

$\boxed{e^{i\frac{\pi}{4}},\ e^{i\frac{3\pi}{4}}}$

in lower half plane

$e^{-i\pi/4},\ e^{-i3\pi/4}$



$\text{Res}(f; e^{i\pi/4}) \qquad \text{Res}(f; e^{i\frac{3\pi}{4}})$

simple poles because $\dfrac{1}{z^4+1} = \dfrac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{-i\pi/4})(z-e^{-i3\pi/4})}$

$f=\dfrac{g}{h}$ $\quad h'(z_0)\neq 0$ $\qquad \| \ \dfrac{g}{h}$ $\quad h'(z)=4z^3$

$\text{Res}(f;z_0) = \dfrac{g(z_0)}{h'(z_0)}$

* $\text{Res}(f; e^{i\pi/4}) = \dfrac{1}{4e^{i(\frac{3\pi}{4})}} = \dfrac{1}{4}e^{-i(\frac{3\pi}{4})} = \dfrac{1}{4}\left(-\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}\right)$ *

** $\text{Res}(f; e^{i\frac{3\pi}{4}}) = \dfrac{1}{4e^{i\frac{9\pi}{4}}} = \dfrac{1}{4}\dfrac{1}{e^{i\pi/4}} = \dfrac{1}{4}e^{-i\pi/4} = \dfrac{1}{4}\left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right)$ **

$\text{Int} = \frac{1}{2}2\pi i\,(* + **)$

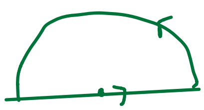
$= \pi i\left(-\frac{2}{4}i\frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\pi$!!

Example (I stole this from the wikipedia page on the Residue Theorem. Also relates to your homework problem 4.3.17) These integrals arise in probability theory: Show that for $b \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(b\,x)}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{\overbrace{e^{i\,b\,x}}^{\cos bx \,+\, i\sin bx}}{x^2+1}\,dx = \pi\, e^{-b}$$

First check that the method two pages back fails for the function you might try first,

- $f(z) = \dfrac{\cos(b\,z)}{z^2+1}$, when $b > 0$....in both the upper and lower half plane.



$$\cos b(x+iy) = \cos(bx+iby) = \frac{1}{2}\left(e^{i(bx+iby)} + e^{-i(bx+iby)}\right)$$

$e^{ibx}e^{-by}$ good in U.H.P bad in L.H.P.

$e^{-ibx}e^{by}$ bad in UHP good in LHP.

$$\left|e^{ib(x+iy)}\right| = \left|e^{ibx}e^{-by}\right| \leq e^{-by} \leq 1$$

in U.H.P.

So thm will apply!

to be continued !!